Planar Near-Field Scanning in the Time Domain, Part 1: Formulation

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Abstract—Time-domain planar near-field measurement techniques are formulated for acoustic and electromagnetic fields. Probe correction is ignored in that it is assumed that the probe measures the exact values of the field on the scan plane. Two fundamentally different approaches are used in deriving three sets of formulas that give the fields in the source-free half space \( z > z_0 \) in terms of their values on the scan plane \( z = z_0 \). In the first approach the time-domain formulas are obtained by inverse Fourier transforming the corresponding frequency-domain formulas. In the second approach the time-domain formulas are derived directly in the time domain by working with time-domain Green’s functions. A companion paper (Part 2) presents the time-domain sampling theorems and computation schemes necessary for the implementation of time-domain planar near-field measurements.

I. INTRODUCTION AND SUMMARY OF RESULTS

For more than 20 years, near-field techniques have been formulated and applied to the measurement of antenna radiation and target scattering [1], [2]. The theory, computer programs and experimental procedures have been successfully developed for the determination of complex radiation and scattering from measurements taken on planar [3], [4], cylindrical [5], [6], and spherical [7]–[9] scanning surfaces in the near field. Since short pulses and wide bandwidths are being used increasingly in radar and communication systems, it has become necessary to accurately determine wide-bandwidth fields of antennas and scatterers. One way of doing this is to use near-field techniques. However, nearly all of the previous work with near-field techniques has been limited to the frequency domain, so radiation or scattering is determined at one frequency at a time. For antennas and scatterers excited by short pulses, it is therefore appropriate to extend the near-field techniques to the time domain.

This paper addresses the problem of formulating planar near-field antenna measurements in the time domain, so that a single set of time-domain near-field measurements yields the entire field and, in particular, the far-field pattern (for one hemisphere) in the time domain or over a wide range of frequencies. The time-domain planar near-field techniques are developed for both acoustic and electromagnetic fields, and the space outside the region occupied by the antenna is assumed to be homogeneous, isotropic, and lossless. Probe correction is ignored, in that the probe is assumed to measure the exact values of the field on the scan plane. The results of this paper first appeared in a Rome Laboratory report [10]. Probe-corrected planar near-field formulas for time-domain acoustic and electromagnetic fields have been derived in a subsequent report [11], and recently submitted for publication [12].

Two fundamentally different approaches are used in deriving time-domain formulas that give the fields in the half space \( z > z_0 \) in terms of their values on the plane \( z = z_0 \). In the first approach the time-domain formulas are obtained by inverse Fourier transforming the corresponding frequency-domain formulas. In the second approach the time-domain near-field formulas are derived directly in the time domain. The equivalence of the resulting time-domain formulas obtained by the two different approaches demonstrates the validity of the formulas and the utility of both approaches.

Section II of this paper presents the frequency-domain planar near-field formulas that are necessary for the subsequent time-domain derivation. The well-known Green’s function formulas [13] and plane-wave spectrum formulas [3] are given for both acoustic and electromagnetic fields. The validity of the plane-wave spectrum formulas is proven, and it is shown that the spectrum in the plane-wave spectrum formulas has one and only one possible singularity for sources in a finite region of space. Far-field expressions are presented both for the plane-wave spectrum formulas and for the Green’s function formulas.

Section III derives three sets of time-domain near-field formulas using the two different approaches mentioned above. The first set of time-domain formulas is obtained by inverse Fourier transforming the frequency-domain Green’s function formulas of Section II to get the corresponding time-domain Green’s function formulas. Next, a time-domain Green’s function is used along with Green’s second identity to derive these time-domain Green’s function formulas directly in the time domain. The formulas for the electromagnetic fields are derived using a time-domain Dirichlet dyadic Green’s function along with the dyadic version of Green’s second identity and agree with those derived by Baum [14]. Far fields are also obtained from these Green’s function formulas, and the far electric field agrees with the result of Hill [15].

The second set of time-domain near-field formulas is obtained by taking the inverse Fourier transform of the frequency-domain plane-wave spectrum formulas. These time-domain formulas give the fields in the half space \( z > z_0 \) in terms of the Radon transform of the fields on the plane \( z = z_0 \), and involve only real fields evaluated at real times. It is shown...
that the Radon transforms of the electric and magnetic fields on the plane \( z = z_0 \) satisfy relations similar to those satisfied by the frequency-domain plane-wave spectra. The time-domain far-field formula obtained from the Radon transform formulas is found to be equivalent to the far-field result obtained from the Green’s function formulas.

Finally, the third set of time-domain formulas are derived in Section III using the analytic Fourier transform in conjunction with the frequency-domain plane-wave spectrum formulas. The resulting time-domain planar near-field formulas are three-dimensional vector analogs to two-dimensional scalar formulas in [16]. These formulas, which also involve the Radon transform, are simpler in form than the formulas of the second set, but they involve analytic fields that cannot be measured directly. To use these analytic formulas to compute the fields directly from time-domain measurements, one ends up converting these formulas to those derived from the standard Fourier transform. Thus, for numerical calculations, the Radon transform formulas derived from the standard Fourier transform and the Green’s function formulas are ultimately the more useful. (The time-domain far-field formulas for both Radon transform formulations are identical, however.) Sampling theorems and computation schemes for calculating time-domain far fields from sampled planar near-field data are presented in a companion paper (Part 2 of this work [17]).

II. SUMMARY OF FREQUENCY-DOMAIN FORMULAS

The planar near-field frequency-domain formulas will be presented in this section for both acoustic and electromagnetic fields. The planar scanning geometry shown in Fig. 1 will be considered. The arbitrary finite source region is located in the half space \( z < z_0 \), and the values of the fields are measured on the plane \( z = z_0 \). The part of space not occupied by the sources is lossless free space with permeability \( \mu \) and permittivity \( \epsilon \). In addition to the rectangular coordinates \( (x, y, z) \), the usual spherical coordinates \( (\rho, \theta, \phi) \) defined by \( x = \rho \cos \phi \sin \theta \), \( y = \rho \sin \phi \sin \theta \), and \( z = \rho \cos \theta \) will be used. Throughout the paper, \( e^{ \text{j} \omega t } \) time dependence is suppressed in all the time-harmonic equations, and in this frequency-domain section it is assumed that the frequency \( \omega \) is positive.

A. Green’s Function Representations

Let us first consider the scalar acoustic field \( \Phi \) satisfying the scalar Helmholtz equation

\[
\nabla^2 \Phi + k^2 \Phi = 0, \quad z > z_0
\]

where \( k \) is the propagation constant and \( \tilde{\mathbf{r}} = x \hat{x} + y \hat{y} + z \hat{z} \). (Then \( \Phi \) can also be any rectangular electric or magnetic-field component.) The finite source region is located in the half space \( z < z_0 \) as shown in Fig. 1.

First consider the scalar acoustic field \( \Phi \) satisfying the Helmholtz equation (1) in the free-space region. Green’s second identity and the radiation condition for the acoustic field and the Dirichlet Green’s function for a half space can be used to show the well-known result [18]

\[
\Phi(\tilde{r}) = \frac{z_o - z}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ikR} \left[ \frac{ik}{R} \right] R \times \tilde{\mathbf{E}}(\tilde{r}_0) \, d\tilde{x}_0 \, d\tilde{y}_0, \quad z > z_0
\]

with \( R = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} \) and \( r_0 = x_0 \hat{x} + y_0 \hat{y} + z_0 \hat{z} \). The formula (2) is the Green’s function representation for the acoustic field \( \Phi \) in the half space \( z > z_0 \) given in terms of \( \Phi \) on the plane \( z = z_0 \).

After having dealt with the acoustic fields, consider the electromagnetic field. Using the dyadic version of Green’s second identity [19, p. 509], the Dirichlet dyadic Green’s function for the half space, and the radiation condition for the electric field, one finds that [10, sec. 2.1.4]

\[
E(\tilde{r}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ikR} \left[ \frac{ik}{R} \right] R \times \tilde{\mathbf{E}}(\tilde{r}_0) \, d\tilde{x}_0 \, d\tilde{y}_0, \quad z > z_0
\]

This is the Green’s function representation of the magnetic field in the half space \( z > z_0 \) given in terms of the \( x \) and \( y \) components of the electric field on the plane \( z = z_0 \).
B. Plane-Wave Spectrum Formulas

In this section we present the plane-wave spectrum formulas that express the acoustic and electromagnetic fields in the half space \( z \geq z_0 \) in terms of their values on the plane \( z = z_0 \). From [3] it follows that the acoustic field in the half space \( z \geq z_0 \) can be expressed as

\[
\Phi(r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T(k_x, k_y) e^{-i(k_y y + k_z z)} dk_x dk_y, \quad z \geq z_0
\]

where \( T \) is the spectrum given by

\[
T(k_x, k_y) = \frac{e^{-i\gamma z}}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(r) e^{-i(k_x x + k_y y)} dz dy, \quad z \geq z_0
\]

with \( \gamma = \left\{ \begin{array}{ll}
v\sqrt{k^2 - k_x^2 - k_y^2}, & k_x^2 + k_y^2 < k^2 \\
i\sqrt{k_x^2 + k_y^2 - k^2}, & k_x^2 + k_y^2 > k^2 \\
\end{array} \right. \) (7)

If \( z \) is set equal to \( z_0 \) in (6), the formula (5) gives the field \( \Phi \) for \( z \geq z_0 \) in terms of \( \Phi \) on the plane \( z = z_0 \). The formulas (5) and (6) comprise the plane-wave spectrum representation of \( \Phi \).

Now consider the electric field \( E \). Since each of its rectangular components satisfies the Helmholtz equation (1) in the free-space region, and since they all are of order \( r^{-1} \) at infinity, the plane wave spectrum representation of the electric field is immediately found from (5) and (6) to be [3]

\[
E(r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T(k_x, k_y) e^{i(k_x x + k_y y + \gamma z)} dk_x dk_y, \quad z \geq z_0
\]

with the spectrum \( T \) for the electric field given by

\[
T(k_x, k_y) = \frac{e^{-i\gamma z}}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(r) e^{-i(k_x x + k_y y)} dz dy, \quad z \geq z_0.
\]

If \( z \) is set equal to \( z_0 \) in (9), the formula (8) gives the electric field for \( z \geq z_0 \) in terms of the electric field on the plane \( z = z_0 \). Since the electric field has zero divergence it is found that the spectrum for the electric field satisfies

\[
T(k_x, k_y) \cdot \mathbf{k} = 0,
\]

where \( \mathbf{k} \) is defined by

\[
\mathbf{k} = k_x \mathbf{x} + k_y \mathbf{y} + \gamma \mathbf{z}.
\]

That is, \( T \) is perpendicular to the propagation direction of the plane wave \( e^{i(k_x x + k_y y + \gamma z)} = e^{i\mathbf{k} \cdot \mathbf{r}} \). Consequently, for \( \gamma \neq 0 \), \( T_z \) can be found from \( T_x \) and \( T_y \), and thus the electric field in the free-space region \( z \geq z_0 \) is determined uniquely by specifying \( E_x \) and \( E_y \) on the measurement plane \( z = z_0 \).

The \( z \) component of the electric field can also be found directly from the \( x \) and \( y \) components by integrating \( \nabla \cdot E = 0 \) with respect to \( z \), to obtain the formula [10, sec. 2.1.1]

\[
E_z(r) = \int_{-\infty}^{z_0} \left[ \frac{\partial}{\partial z} E_x(x, y, z') \right] dz', \quad z \geq z_0.
\]

Using the induction law \( \nabla \times E = i\omega \mu \mathbf{H} \), which gives the magnetic field in terms of the electric field, it is found that the magnetic field is given by (8) with \( E \) replaced with \( H \) and \( T \) replaced by

\[
T_H(k_x, k_y) = \frac{\sqrt{\mu}}{v} \mathbf{k} \times T(k_x, k_y).
\]

Because the divergence of the magnetic field is zero in the free-space region, the spectrum for the magnetic field also satisfies (10) (as is seen directly from (13)). As with the electric field, the tangential components of the magnetic field on the plane \( z = z_0 \) determine uniquely all the field components in the half space \( z \geq z_0 \). Also, \( H_z \) can be determined from \( H_x \) and \( H_y \) in the half space \( z \geq z_0 \) by the formula (12), with \( E \) replaced by \( H \). In general, any two of the six components of \( E \) and \( H \) on the plane \( z = z_0 \) determine all fields in the half space \( z > z_0 \) [10, sec. 2.1.1].

In principle, once the spectrum is computed from the measured data on the plane \( z = z_0 \), (8) and (13) can be used to compute the electric and magnetic fields on any plane \( z \) (not just \( z \geq z_0 \)) up to the source region. In practice, however, the scan plane \( z = z_0 \) is usually chosen a few wavelengths away from the test antenna to keep multiple reflections between the probe and test antenna at an acceptably low level. This means that most of the evanescent fields have decayed to too small a value to be detected by the measurement system. Thus, most of the evanescent part of the spectrum could not be computed with any accuracy from the measured data in (9) on the plane \( z = z_0 \), and the evanescent fields, which may dominate in a region within a few wavelengths from the source, could not be computed from (8).

Let us now discuss the validity of the plane-wave spectrum formulas presented in this section and the possible singularities of the spectrum. Start with the acoustic field and note that this field is neither absolutely integrable nor square integrable on any plane \( z = z_0 \). This follows from the fact that \( \Phi(x, y, z_0) = O((x^2 + y^2)^{-1/2}) \) as \( (x^2 + y^2)^{1/2} \rightarrow \infty \) so that both the integrals \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\Phi(x, y, z)| dx dy \) and \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\Phi(x, y, z)|^2 dx dy \) are infinite. Consequently, the theory of the Fourier transform in the Lebesgue spaces \( L_1 \) or \( L_2 \) [21], [22] cannot be used to prove the validity of (5) and (6) because the acoustic field belongs neither to \( L_1 \) nor \( L_2 \). (Note that the energy radiated through the plane \( z = z_0 \) is proportional to the integral of \( \Phi^2 \) over this plane [23]. This energy is finite even though the integral of \( |\Phi|^2 \) over the plane \( z = z_0 \) is infinite.)

In [10, sec. 2.1.2] we have used Green's second identity and the radiation condition for the acoustic field to show that (5) and (6) indeed produce the correct acoustic field if the integral
in the expression (6) for the spectrum $T$ is calculated in polar coordinates with the angular integration performed first. Alternatively, this integral could be calculated in rectangular coordinates by first evaluating the double integral over a finite rectangle with a fixed sidelength ratio and then letting the sidelengths of the rectangle approach infinity. These two ways of calculating the spectrum are exactly the two ways used for most practical planar near-field measurements.

In [10, sec. 2.1.1.3] it is shown that the spectrum $T(k_x, k_y)$ is a continuous bounded function of $k_x$ and $k_y$ except possibly at $\gamma = 0$ and that the spectrum may be written as

$$T(k_x, k_y) = \frac{i}{\gamma} F(k_x, k_y)$$

(14)

for all $(k_x, k_y)$, where $F$ is a continuous bounded function for all $(k_x, k_y)$. Furthermore, in [10, App. A] it is shown that the function $F$ is infinitely differentiable for all $(k_x, k_y)$ except possibly at $\gamma = 0$ and that $F$ is an infinitely differentiable function of $(\gamma, k_y)$, except at $\gamma = k$ when the propagation vector is expressed in polar coordinates.

For the electromagnetic field one finds similarly that the plane-wave spectrum formulas are correct provided that the spectrum $\tilde{T}$ is calculated in the above prescribed manner and that the electromagnetic spectrum may be written in the form (14). The $\gamma^{-1}$ singularity in the spectrum $\tilde{T}$ is also exhibited by the formulas in Kerns [3, (2.2.7a), (2.2.7b)], relating spectra to the prescribed currents.

C. Frequency-Domain Far Fields

The far-field formulas that give the fields as $r \to \infty$ will be presented in this section in terms of the fields on the plane $z = z_0$.

Start with the acoustic field $\Phi$ given by (2) and define the far-field pattern $\mathcal{F}(\theta, \phi)$ such that the acoustic far field is given by $\Phi(\bar{r}) \sim \mathcal{F}(\theta, \phi)e^{ikr_{\bar{r}}}$. Inserting the far-field approximations $\bar{R} = r - (x_0 \cos \theta \sin \phi + y_0 \sin \theta \cos \phi + z_0 \cos \theta) + O(r^{-1})$ and $2\bar{z}/R = -\cos \theta + O(r^{-1})$ into the Green’s function representation (2), one finds that the acoustic far-field pattern is

$$\mathcal{F}(\theta, \phi) = -\frac{ik \cos \theta}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Phi(\bar{r}) e^{-ikr_{\bar{r}}} dz_0 dy_0, \quad z > z_0$$

(15)

with $\bar{r}_0 = x_0 \bar{x} + y_0 \bar{y} + z_0 \bar{z}$, and $\bar{r} = \bar{x} \cos \phi \sin \theta + \bar{y} \sin \phi \sin \theta + \bar{z} \cos \theta$. Note that this may also be written in terms of the spectrum $T$ in (6) as

$$\mathcal{F}(\theta, \phi) = -\frac{ik \cos \theta}{\mu} T(k \cos \phi \sin \theta, k \sin \phi \sin \theta), \quad z > z_0.$$  

(16)

Similarly, define a far-field pattern $\mathcal{F}(\theta, \phi)$ for the electric field such that the far electric field is $\bar{E}(\bar{r}) \sim \mathcal{F}(\theta, \phi)e^{ikr_{\bar{r}}}$. Then use the fact that $\bar{R} = \bar{r} + O(r^{-1})$ along with the previously used asymptotic relations in the Green’s function representation (3) for the electric field to get the following formula for the electric far-field pattern

$$\mathcal{F}(\theta, \phi) = \frac{i k}{2\pi} \bar{r} \times \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \bar{z} \times \bar{E}(\bar{r}_0) e^{-ikr_{\bar{r}}} dz_0 dy_0, \quad z > z_0.$$  

(17)

Inserting the relation $\bar{r} \times (\bar{z} \times \bar{E}) = (\bar{r} \cdot \bar{E}) \bar{z} - \cos \theta \bar{E}$, and the definition of the plane-wave spectrum from (9) into (17) yields

$$\tilde{\mathcal{F}}(\theta, \phi) = -ik \cos \theta \bar{T}(k \cos \phi \sin \theta, k \sin \phi \sin \theta), \quad z > z_0$$

(18)

where we have made use of (10), that is, $\bar{r} \cdot \bar{T}(x, y) = 0$.

The far magnetic field is similarly written as $\tilde{H}(\bar{r}) \sim \mathcal{F}(\theta, \phi)e^{ikr_{\bar{r}}} / r$ and we find that the magnetic far-field pattern is given by

$$\mathcal{F}(\theta, \phi) = \frac{1}{\mu} \frac{\mu}{2\pi} \bar{r} \times \bar{E}(\bar{r}_0) e^{-ikr_{\bar{r}}} dz_0 dy_0, \quad z > z_0.$$  

(19)

or alternatively

$$\mathcal{F}(\theta, \phi) = \frac{1}{\mu} \frac{\mu}{2\pi} \bar{r} \times \bar{E}(\bar{r}_0) e^{-ikr_{\bar{r}}} dz_0 dy_0, \quad z > z_0.$$  

(20)

It should be noted that the derivation of the formulas (15), (17), and (20) required that the limit as $r \to \infty$ be interchanged with the infinite $(x_0, y_0)$ integration. We cannot use standard theorems of calculus to justify this interchange because $\Phi$ and $\bar{E}$ are not absolutely integrable on the infinite $(x_0, y_0)$ plane. However, these far-field formulas can be proven valid by first expressing $\bar{E}(\bar{r}_0)$ and $\bar{E}(\bar{r}_0)$ in the integrals of (15), (17), and (20) in terms of their volume sources, or equivalent surface sources, and then proceeding as in [10, sec. 2.1.2].

III. Time-Domain Formulas

This section derives the time-domain planar near-field formulas for both acoustic and electromagnetic fields. The time-domain analogs to the Green’s function frequency-domain formulas of Section II-A are derived in Part A of this section first by using the frequency-domain formulas and the Fourier transform. Then they are derived by working directly in the time domain with a time-domain Green’s function. Furthermore, Part A derives a formula that gives the time-domain magnetic field in the half space $z > z_0$ in terms of the time-domain electric field on the plane $z = z_0$ and discusses the class of time functions for which the Fourier transform can be used to calculate the time-domain magnetic field.

As in the previous section, the planar scanning geometry is shown in Fig. 1 with the finite source region located in the half space $z < z_0$. The fields are specified on the plane $z = z_0$, and we are interested in calculating the fields in the half space $z > z_0$. All fields are assumed to be zero for $t < 0$, and the time dependence of the fields for $t \geq 0$ is assumed to be such that their frequency spectrum is finite at $\omega = 0$. This assumption, which is equivalent to requiring that the time integral over all time of the fields be finite, eliminates possible static fields. Planar near-field formulas for static electric and magnetic fields in various types of lossy materials are derived in [10, sec. 2.5].
The acoustic field $\Phi$ in the lossless, source-free, and homogeneous half space $z \geq z_0$ satisfies the scalar wave equation

$$\nabla^2 \Phi(r, t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \Phi(r, t) = 0, \quad z > z_0$$

(21)

where $c$ is the acoustic speed. Similarly, the electromagnetic fields in the source-free half space $z > z_0$ satisfy the time-dependent Maxwell equations for a lossless medium

$$\nabla \times \vec{H} = c \frac{\partial}{\partial t} \vec{E}, \quad \nabla \times \vec{E} = \mu_0 \frac{\partial}{\partial t} \vec{H}, \quad \nabla \cdot \vec{H} = 0, \quad \nabla \cdot \vec{E} = 0$$

(22)

where $\varepsilon$ and $\mu$ are the space and time independent permittivity and permeability, respectively, which implies that the electric field satisfies

$$\nabla^2 \vec{E}(r, t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{E}(r, t) = 0, \quad z > z_0$$

(23)

where $c = (\varepsilon \mu)^{-1/2}$ is the speed of light. In this time-domain section, all time-harmonic fields are labeled with subscript $\omega$.

**A. Time-Domain Green's Function Formulas**

The formulas of this section are time-domain analogs to the frequency-domain Green's function formulas of Section II-A. These time-domain formulas are first obtained by Fourier transforming the frequency-domain Green's function formulas and are then rederived by working directly in the time domain with time-domain Green's functions.

**Derivation from the Fourier transform:** Begin by expressing the acoustic field $\Phi$, satisfying the wave equation (21), in terms of the Fourier integral

$$\Phi(r, t) = \int_{-\infty}^{\infty} \Phi_\omega(r)e^{-i\omega t}d\omega$$

(24)

and write the propagation constant as $k = \omega/c$ in the frequency-domain formula (2) to get

$$\Phi_\omega(r) = \frac{z_0 - z}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_\omega(\hat{r}_0) \left[ i\frac{\omega}{c} - \frac{1}{R} \right] e^{i\omega R/c} dx_0 dy_0, \quad z > z_0$$

(25)

with $\hat{r}_0 = z_0 \hat{x} + \hat{y} \hat{y} + z_0 \hat{z}$ and $R = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$. Inserting $\Phi_\omega$ from (25) into (24) and interchanging the orders of integration, (24) becomes

$$\Phi(r, t) = \frac{z - z_0}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{R^2} \left[ \frac{\partial \Phi}{\partial t}(\hat{r}_0, t - R/c) + c \frac{\Phi}{R} \right] dx_0 dy_0, \quad z > z_0$$

(26)

where the derivative rule

$$\frac{\partial \Phi}{\partial t}(\hat{r}, t) = \int_{-\infty}^{\infty} -i\omega \Phi_\omega(\hat{r})e^{-i\omega t}d\omega$$

(27)

has been used. Equation (26) gives the time-domain field in the region $z > z_0$ in terms of the field on the plane $z = z_0$.

Similarly, by Fourier transforming the electromagnetic frequency-domain formula (3) one finds that the time-domain electric field is given by [14]

$$E(r, t) = \frac{-1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{R}{R^2} \times \left[ \frac{1}{c^2} \frac{\partial}{\partial t} \vec{E}(\hat{r}_0, t - R/c) + \frac{1}{R} \vec{E}(\hat{r}_0, t - R/c) \right] dx_0 dy_0, \quad z > z_0.$$  

(28)

Although it has been assumed that all fields are zero for $t < 0$, the formula (28) can also be shown valid for static electric fields [10, sec. 3.1.1]. The corresponding formulas for the magnetic field will be derived later in this section.

In this section it has been assumed that the time dependence of the field is such that it's time Fourier transform exists. For time signals with infinite energy, such as the unit step function, the standard Fourier transform cannot be used because the Fourier integral does not exist and the analysis in this section does not apply. However, one can repair this deficiency in the derivation of (28) and of (77) below in a number of ways. One way is to use distribution theory. Another way is to give the time signal with infinite energy an exponential decay and then let the decay go to zero after having derived the near-field formulas. Still another way is to use the complex Fourier transform, described in Dettman [24, pp. 365-369], which is obtained by letting the real contour from the standard Fourier integration become complex [10, sec. 3.1.3].

**Derivation from time-domain Green's functions:** This section derives the time-domain near-field transformation formula (28) for the electric field using a time-domain Green's function approach. The idea to this approach comes from Stratton [25, sec. 8.1.1, who deals only with scalar fields, and is based on Green's second identity. The derivation of the acoustic formula (26) can be found in [10, sec. 3.1.2] and is much simpler than the electromagnetic derivation presented here.

To derive the near-field formula (28) for the electric field directly in the time domain, we shall make use of the time-domain Dirichlet dyadic Green's function $G_D(\hat{r}, \hat{r}', t, t')$ for the half space $z > z_0$. This function can be found by Fourier transforming the corresponding frequency-domain dyadic Green's function [26, p. 68] and is given by

$$G_D(\hat{r}, \hat{r}', t, t') = \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{R^2} \nabla \times \nabla' \Phi_\omega(\hat{r}_0, \hat{r}_0', 0) du' du + 2i R_G(\hat{r}, \hat{r}', t, t')$$

(29)

where

$$G_D(\hat{r}, \hat{r}', t, t') = G(\hat{r}, \hat{r}', t, t') - G(\hat{r}, \hat{r}', t, t')$$

(30)

is the scalar Dirichlet Green's function for the half space $z \geq z_0$ and

$$G(\hat{r}, \hat{r}', t, t') = \frac{\delta(t - t' + R/c)}{4\pi R}$$

(31)

is the free-space Green's function. This Green's function is retarded with respect to the primed variables. The dyadic
Green’s function (29) satisfies the dyadic wave equation
\[
\nabla \times \nabla \times \overline{G}_D + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \overline{G}_D = \delta(\vec{r} - \vec{r}') \delta(t - t'), \quad z > z_0 \tag{32}
\]
and the Dirichlet boundary condition \(\vec{z} \times \overline{G}_D = 0\) on the plane \(z = z_0\). The dyadic version of Green’s second identity [19, p. 509] will now be used to derive a time-domain near-field formula for the electric field. Employing this identity to the region bounded by the circular disk \(C(R_0)\) and the half sphere \(S(R_0)\) in Fig. 2, the identity
\[
\nabla \times \nabla \times \overline{E} \cdot \overline{G}_D - \overline{E} \cdot \nabla \times \nabla \times \overline{G}_D = \frac{1}{c^2} \frac{\partial}{\partial t} \left[ \frac{\partial}{\partial t} \overline{E} \right]
\]
and the Dirichlet boundary condition for \(\overline{G}_D\), one gets
\[
-\overline{E}(\vec{r}', t') \delta(t - t') = \frac{1}{c^2} \frac{\partial}{\partial t} \int_{\int \Omega(\overline{S}(R_0) \cup C(R_0))} \left[ \left( \frac{\partial}{\partial t} \overline{E} \right) \cdot \overline{G}_D - \overline{E} \cdot \frac{\partial}{\partial t} \overline{G}_D \right] dV
\]
\[
- \int_{\int \Omega(C(R_0))} \left[ -\overline{\vec{z}} \times \nabla \times \overline{G}_D \right] dS,
\]
where \(R_0\) is assumed to be sufficiently large so that the causal field is zero on \(S(R_0)\). The identity \(\nabla \times \overline{G}_D = 2\overline{G} \times \overline{\vec{I}}\) for \(z = z_0\), along with [19, pp. 507-508, (15), (19)]
\[
\overline{E} \cdot \overline{\vec{z}} \times \nabla \times \overline{\vec{I}} = -\left( \overline{\vec{z}} \times \overline{E} \right) \cdot \left( \nabla \times \overline{\vec{G}} \right) = -\nabla \times \left( \overline{\vec{z}} \times \overline{E} \right)
\]
show that \(\overline{E} \cdot \left[ -\overline{\vec{z}} \times \nabla \times \overline{G}_D \right] = 2\overline{\nabla} \times \left( \overline{\vec{z}} \times \overline{E} \right)\) where \(G\) is the time domain free-space Green’s function (31). Inserting this into (34), integrating (34) with respect to \(t\) from \(-\infty\) to \(+\infty\), and letting \(R \to \infty\), one finds that
\[
\overline{E}(\vec{r}', t') = 2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \nabla' G(\vec{r}', \vec{r}', \vec{r}, t')
\]
\[
\times \left[ \overline{\vec{z}} \times \overline{E}(\vec{r'}, t) \right] dt'dxdy', \quad z' > z_0
\]
where the facts that \(\lim_{t+\to+\infty} \overline{G}_D = 0\) and \(\lim_{t-\to-\infty} \frac{\partial}{\partial t} \overline{G}_D = 0\) have been used to show that there is no contribution from the first integral in (34). Now, from (31),
\[
\nabla' G(\vec{r}', \vec{r}', \vec{r}, t') = R' \frac{1}{4\pi \epsilon R^2}
\]
\[
\left[ \delta'(t - t' + R/c) - \frac{c}{R} \delta(t - t' + R/c) \right]
\]
where \(R' = \vec{r}' - \vec{r}\) and \(\delta'(t)\) is the derivative of the delta function. Inserting this result into (36) and performing the change of variables \(\vec{r} \to \vec{r}_0, \vec{r}' \to \vec{r}',\) and \(t \to t'\), (36) becomes
\[
\overline{E}(\vec{r}, t) = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{c^2} \frac{\partial}{\partial t} \overline{E}(\vec{r}_0, t - R/c)
\]
\[
+ \frac{c}{R^2} \left[ \overline{\vec{z}} \times \overline{E}(\vec{r}_0, t - R/c) \right] dx_0 dy_0 dz_0, \quad z > z_0
\]
which is seen to agree with the near-field formula (28), obtained by Fourier transforming the frequency-domain formula (3).
$z > z_0$ in terms of the $x$ and $y$ components of the electric field on the plane $z = z_0$. Both this formula for the magnetic field and formula (28) for the electric field were derived previously by Baum [14], [27] by taking the inverse Fourier transform of the Smythe formulas [20].

The formula (39) could have been derived directly in the time domain in the following way. Use Maxwell's equations and the initial conditions $E(r, t) = 0$, $H(r, t) = 0$ for $t < 0$ to prove that $H(r, t) = -\frac{1}{c} \int_0^t \nabla \times E(r, t') dt'$. Insert the time-domain formula (38) for the electric field into this expression for the magnetic field, and (39) is recovered. Alternatively, (39) can be derived in a manner similar to the derivation of the electric field formula (38), using the Neumann dyadic Green's function for the half space $t > 0$, close a switch that charges a conductor and causes a magnetic field to remain constant for all time ($-\infty < t < \infty$), unlike (38), which remains valid for electrostatic fields. If such a magnetostatic field is present, one has to add its value to the formula (39) to get the correct total magnetic field.

The discussion of static fields brings up a subtlety in (38) and (39) that may be easily overlooked. Suppose all sources and fields are zero for $t < 0$ so that (38) and (39) apply. At $t = 0$, close a switch that charges a conductor and causes current to flow through a resistor. After a time $t_0$, assume all charge and current have practically reached a constant value, so that for $r < c(t - t_0)$ there remain only static electric and magnetic fields. These remnant static electric and magnetic fields, which build up from zero fields at $t = 0$, are entirely determined from the formulas (38) and (39), respectively. This is unremarkable for (38), which, as mentioned before, remains valid for electrostatic fields ($-\infty < t < \infty$) but not for (39), which represents a remnant magnetostatic field for $t > 0$ even though it cannot represent a purely magnetostatic field for all time ($-\infty < t < \infty$).

For the starting point of the derivation is the expression (6) for the frequency-domain plane-wave spectrum, rewritten as

$$T_\omega(\omega_\xi, \omega_\eta) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Phi_\omega(f_0) e^{-i\omega(\xi_0 + \eta_0)} d\xi_0 d\eta_0, \quad z \geq 0 \quad (41)$$

where $f_0 = \hat{\alpha}x_0 + \hat{\beta}y_0$. Let $T(\xi, \eta, t)$ be the inverse Fourier transform of $T_\omega(\omega_\xi, \omega_\eta)$ given by

$$T(\xi, \eta, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Phi_\omega(f_0) e^{-i\omega(f_0 + \xi + \eta)} d\omega d\xi_0 d\eta_0$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Phi_\omega(f_0, t + \xi x_0 + \eta y_0) d\xi_0 d\eta_0 \quad (42)$$

which involves only a real integral. Thus for this kind of time dependence of the electric field one can compute its time integral directly in terms of its Fourier transform evaluated at real frequencies.

### B. Time-Domain Analogs of the Plane-Wave Spectrum Formulas

We will now derive formulas that are time-domain analogs to the frequency-domain plane-wave spectrum formulas of Section II-B. Without loss of generality, it is assumed that $z_0 = 0$ because it substantially simplifies the analysis of this section.

Formulas obtained from the standard Fourier transform: The starting point of the derivation is the expression (6) for the frequency-domain plane-wave spectrum, rewritten as

$$\tau_{\omega}(\omega_\xi, \omega_\eta) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Phi_\omega(f_0) e^{-i\omega(\xi_0 + \eta_0)} d\xi_0 d\eta_0, \quad z \geq 0 \quad (41)$$

where $f_0 = \hat{\alpha}x_0 + \hat{\beta}y_0$. Let $T(\xi, \eta, t)$ be the inverse Fourier transform of $T_\omega(\omega_\xi, \omega_\eta)$ given by

$$T(\xi, \eta, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Phi_\omega(f_0) e^{-i\omega(f_0 + \xi + \eta)} d\omega d\xi_0 d\eta_0$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Phi_\omega(f_0, t + \xi x_0 + \eta y_0) d\xi_0 d\eta_0 \quad (42)$$

which is seen to equal the Radon transform [28] of the time-domain field $\Phi$ on the plane $z = 0$.

Writing the frequency-domain plane-wave spectrum formula (5) in terms of the function $T_\omega(\omega_\xi, \omega_\eta)$ and introducing $(k_x, k_y) = (\omega_\xi, \omega_\eta)$, one finds for $\omega > 0$ that

$$\Phi_\omega(\vec{r}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \omega^2 T_\omega(\omega_\xi, \omega_\eta) e^{i\omega(\xi x_0 + \eta y_0 + \zeta)} d\xi_0 d\eta_0, \quad z \geq 0 \quad (43)$$

where we have written $\gamma = \omega \xi_0$ and introduced the new spectral variable $\zeta$ by

$$\zeta = \left\{ \begin{array}{ll} \sqrt{\xi^2 - \xi^2 - \eta^2}, & \xi^2 + \eta^2 < \epsilon^{-2} \\ i\sqrt{\xi^2 + \eta^2 - \epsilon^{-4}}, & \xi^2 + \eta^2 > \epsilon^{-2}. \end{array} \right. \quad (44)$$
The fact that the time-domain fields are real shows that \( \Phi^*_\omega = \Phi_\omega \), which in turn shows that \( \Phi_\omega \) for \( \omega < 0 \) is given by (43), with \( \zeta \) replaced by \( \zeta^* \) (here \( \ast \) indicates complex conjugation). This enables us to give the following expression for \( \Phi_\omega \), valid for all real \( \omega > 0 \):

\[
\Phi_\omega(r) = \frac{1}{2\pi} \int_{\xi_2 + \eta^2 > -c^2} \int_{\xi_2 + \eta^2 < -c^2} \omega^2 T_\omega(\omega \xi, \omega \eta)e^{i(\xi x + \eta y + c\zeta)} d\xi d\eta
+ \frac{1}{2\pi} \int_{\xi_2 + \eta^2 > -c^2} \int_{\xi_2 + \eta^2 < -c^2} \omega^2 T_\omega(\omega \xi, \omega \eta)e^{i(\xi x + \eta y)}
\cdot e^{-i|\xi|d\xi d\eta}, \quad z \geq 0.
\]

(45)

Taking the inverse Fourier transform of (45) and using the convolution rule \( \int_{-\infty}^{\infty} f_g \omega e^{-i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t')g(t-t')dtdt' \) in the last integral, one finds that the time-domain field in the half space \( z > 0 \) is given by

\[
\Phi(r, t) = -\frac{1}{2\pi} \int_{\xi_2 + \eta^2 < -c^2} \frac{\partial^2}{\partial t^2} T(\xi, \eta, t - \xi x - \eta y - \zeta z) d\xi d\eta - \frac{1}{2\pi} \int_{\xi_2 + \eta^2 > -c^2} \int_{-\infty}^{\infty} \frac{\partial^2}{\partial t^2} T(\xi, \eta, t - \xi x - \eta y)
\cdot \frac{\xi z}{|\xi|^2 z^2 + t^2} d\xi d\eta d\xi d\eta, \quad z > 0
\]

(46)

where the identity

\[
\int_{-\infty}^{\infty} e^{-i|\xi|d\xi} e^{-i\omega t} d\omega = \frac{2|\xi|}{|\xi|^2 z^2 + t^2}, \quad z > 0
\]

(47)

has been used.

Note that (46) gives the field in the half space \( z > 0 \) in terms of the Radon transform of the field on the plane \( z = 0 \). Note also that the second time derivative of the Radon transform, and therefore also the second time derivative of the field on the plane \( z = 0 \), is required to calculate the field for \( z > 0 \). The formula (46) is a time-domain analog to the frequency-domain plane-wave spectrum formula (5). The formulas derived in this section hold for each of the rectangular components of the electric and magnetic fields, and one finds from (46) that the space component of \( T(\xi, \eta, t) \) can be found in terms of the \( x \) and \( y \) components of \( T(\xi, \eta, t) \) for \( \xi^2 + \eta^2 > c^{-2} \) by taking the inverse Hilbert transform of (55), or by multiplying (53) by \( |\omega|/\omega \) and taking the Fourier transform to get

\[
T_z(\xi, \eta, t) = \mathcal{H}^*T(\xi, \eta, t) \cdot (\xi \xi + \eta \eta)/|\xi|, \quad \xi^2 + \eta^2 > c^{-2}.
\]

(56)

The minus sign on the right side of (54) occurs because we are dealing with analyticity in the lower half of the complex \( t \) plane rather than in the upper half as in Morse and Feshbach [29]. Sufficient conditions for the existence of the Hilbert transform of \( f(t) \) are that \( f(t) \) be Hölder continuous and that \( f(t) \) approaches the definite value \( f(\infty) \) as \( t \to \pm \infty \) such that \( |f(t) - f(\infty)| \leq C/|t|^\alpha \), where \( C \) and \( \alpha \) are positive constants [30, sec. 43]. This latter condition plays the role of the Hölder condition at \( t = \pm \infty \) and, of course, \( f(\infty) = 0 \) for the functions of time we are considering that are zero for \( t < 0 \). The Hilbert-transform relationships in (54) can also be proven by first multiplying the integrand of the first integral in (54) by \( e^{-i|\omega|/\omega} (\alpha > 0) \), secondly applying the convolution theorem, thirdly expressing the convolution as a principal-value integral, and finally taking the limit of the integral as \( \alpha \to 0 \) by means...
of the theorem in Hobson \[31, \text{sec. 2251}\.) Equations (52) and (56) enables us to calculate the $z$ component of the Radon transform of the electric field from its $x$ and $y$ components.

The formulas derived in this section hold also for the magnetic field provided that $E$ is replaced by $H$, where $T_H$ is given by (49) with $E$ replaced by $H$. The relation (13) between the spectrum $T_H$ for the electric field and the spectrum $T_{H,\omega}$ for the magnetic field shows that

$$T_{H,\omega}(\omega, \eta) = \frac{1}{\mu} (\xi \hat{x} + \eta \hat{y} + \zeta \hat{z}) \times T_{H}(\omega, \eta), \quad \omega > 0$$

and the same result with $\zeta$ replaced by $\zeta^*$ for $\omega < 0$. Again, for $\xi^2 + \eta^2 < c^{-2}$, (57) shows that

$$T_H(\xi, \eta, t) = \frac{1}{\mu} (\xi \hat{x} + \eta \hat{y}) \times \mathbf{T}(\xi, \eta, t), \quad \xi^2 + \eta^2 < c^{-2}$$

and for $\xi^2 + \eta^2 > c^{-2}$,

$$T_H(\xi, \eta, t) = \frac{1}{\mu} (\xi \hat{x} + \eta \hat{y}) \times \mathbf{T}(\xi, \eta, t) + \frac{\zeta}{\mu} \hat{z} \times \mathbf{H}(\xi, \eta, t), \quad \xi^2 + \eta^2 > c^{-2}$$

Thus the Radon transform of the magnetic field on the plane $z = 0$ is determined from the Radon transform of the electric field on that plane.

The formulas shown in this section are time-domain analogs to the plane-wave spectrum frequency-domain formulas of Section II-B. It is seen that the Radon transform in the time-domain formulas plays the role of the spatial Fourier transform in the frequency-domain formulas. Furthermore, the Radon transform of the electric and magnetic fields on the plane $z = 0$ satisfy the relations (52)-(55) and (58)-(59), which are similar to but more complicated than the relations (10) and (13) satisfied by the plane-wave spectra for the electric and magnetic fields. If the evanescent fields are negligible, the second integrations in (46) and (50) are negligible and the time-domain Radon-transform relations take the simpler form of the analogous frequency-domain Fourier transform relations.

Formulas obtained from the analytic Fourier transform: In this section it will be shown that by proceeding as above with the analytic Fourier transform \[16\] instead of the standard Fourier transform one may obtain formulas that are in a simpler form than those above. However, the formulas of the present section are valid only for the so-called analytic field, which is defined below and cannot be determined directly from measurements.

The analytic field, which corresponds to the analytic Fourier transform, is denoted by $\Phi^\alpha$ and is given by the inverse analytic Fourier transform

$$\Phi^\alpha(\vec{r}, t) = 2 \int_0^{\infty} \Phi_\omega(\vec{r}) e^{-i\omega t} d\omega, \quad \text{Im}(\omega) \leq 0$$

where $\Phi_\omega$ is the usual Fourier transform of $\Phi$ given by

$$\Phi_\omega(r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(\vec{r}, t) e^{i \omega t} dt.$$ \hfill (61)

The relation between the real field $\Phi$ and analytic field $\Phi^\alpha$ is \[29, \text{pp. 370-372}\], \[16\]

$$\Phi^\alpha(\vec{r}, t) = \Phi(\vec{r}, t) + i \mathcal{H} \Phi(\vec{r}, t), \quad \text{t real} \hfill (62)$$

where $\mathcal{H} \Phi(\vec{r}, t)$ is the Hilbert transform of $\Phi(\vec{r}, t)$:

$$\mathcal{H} \Phi(\vec{r}, t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\Phi(\vec{r}, t')}{t - t'} dt', \quad \text{Im}(\omega) \leq 0.$$ \hfill (63)

Again, the minus sign in (62) occurs because $\Phi^\alpha$ is analytic in the “lower” half plane. The real field is thus recovered from the analytic field by the equation $\Phi(\vec{r}, t) = \mathcal{H} \Phi^\alpha(\vec{r}, t)$ for real $t$. From the definition (60) of the analytic field it follows that $\Phi^\alpha$ is analytic in the lower half of the complex $t$ plane, that is, in the region $\text{Im}(t) \leq 0$.

For $\text{Im}(t) < 0$, the analytic field is given in terms of the real field by the formula

$$\Phi^\alpha(\vec{r}, t) = \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{\Phi(\vec{r}, t)}{t - t'} dt', \quad \text{Im}(t) < 0.$$ \hfill (64)

(This formula can be proved by inserting $\Phi^\alpha(\vec{r})$ from (61) into (60), interchanging the order of integration, and performing the integration over $\omega$.)

One can now define the analytic field $T^\alpha(\xi, \eta, t)$, which corresponds to $T(\xi, \eta, t)$ defined in (42), by

$$T^\alpha(\xi, \eta, t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi^\alpha(\vec{r}, t) e^{-i(\xi x + \eta y + \zeta z)} d\omega d\zeta d\eta$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi^\alpha(\vec{r}_0, t + \xi x_0 + \eta y_0) dx_0 dy_0, \quad \text{Im}(t) \leq 0.$$ \hfill (65)

which is seen to equal the Radon transform of $\Phi^\alpha$ on the plane $z = 0$.

Taking the inverse analytic Fourier transform of the frequency-domain formula (43), one finds that the analytic field in the half space $z > 0$ is given by

$$\Phi^\alpha(\vec{r}, t) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{j\beta}{\xi^2 + \eta^2 + c^2}$$

$$T^\alpha(\xi, \eta, t - \xi x - \eta y - \zeta z) d\xi d\eta, \quad z > 0, \quad \text{Im}(t) \leq 0.$$ \hfill (66)

Note that (66) gives the analytic field in the half space $z > 0$ in terms of the Radon transform of the analytic field on the plane $z = 0$. Note also that this formula is much simpler than the formula (46) obtained from the standard Fourier transform.

To calculate the real field $\Phi = \mathcal{H} \Phi^\alpha$ for real times from (66), one must know the Radon transform $T^\alpha$ for complex times because $\zeta$ becomes complex when $\xi^2 + \eta^2 > c^{-2}$. Equation (65) shows that to get $T^\alpha$ for complex times, the analytic field $\Phi^\alpha$ must be known on the plane $z = 0$ for complex times. Consequently, to use (66) to calculate the real field $\Phi$ for real times one must know the analytic field $\Phi^\alpha$ on the plane $z = 0$ for both complex and real times. Substituting $\Phi^\alpha(\vec{r}, t)$ from (64) into (65) shows that $T^\alpha(\xi, \eta, t)$ also obeys the relationship

$$T^\alpha(\xi, \eta, t) = \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{T(\xi, \eta, t - \xi x - \eta y - \zeta z) \xi d\xi}{t - t'}.$$ \hfill (67)

Furthermore, substitution of $T^\alpha$ from (67) into (66) and taking the real part to get $\Phi(\vec{r}, t)$ produces the formula (46), which was derived from the standard Fourier transform.
To derive the corresponding formulas for the electromagnetic field, start by defining

\[ \bar{T}^a(\zeta, \eta, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \bar{E}^a(r_0, t + \xi x_0 + \eta y_0) dx_0 dy_0, \quad \text{Im}(t) \leq 0 \]

which is the Radon transform of the analytic electric field \( \bar{E}^a(r, t) \).

From (66), it is then found that the time-domain electric field on the plane \( z = 0 \), where

\[ \bar{E}^a(r, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \bar{E}(r, t) e^{-i\omega t} d\omega, \quad \text{Im}(t) \leq 0 \]

(Fig. 70)

From (66), it is then found that the time-domain electric field is given by the Radon transform

\[ \bar{E}^a(r, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\partial^2}{\partial t^2} \bar{T}^a(\xi, \eta, t - \xi x_0 - \eta y_0) d\xi d\eta, \]

\[ z > 0, \quad \text{Im}(t) \leq 0. \quad (71) \]

For \( \text{Im}(t) < 0 \), the analytic field and its Radon transform can be written as in (64) and (67), namely

\[ \bar{E}^a(r, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \bar{E}(r, t') d\omega', \quad \text{Im}(t) < 0 \]

(72)

From (66), it is then found that the time-domain electric field is given by the Radon transform

\[ \bar{T}^a(\zeta, \eta, t) = \frac{1}{\mu} \int_{-\infty}^{+\infty} \bar{T}^a(\xi, \eta, t - \xi x_0 - \eta y_0) d\xi d\eta, \quad \text{Im}(t) < 0. \quad (73) \]

Substitution of \( \bar{T}^a \) from (73) into (71) and taking the real part to get \( \bar{E}(r, t) \) yields the standard Fourier-transform formula (50).

The formula (51), along with the analytic Fourier transform (which only involves positive \( \omega \)), shows that in agreement with (52) and (55)

\[ \bar{T}^a(\zeta, \eta, t) \cdot (\xi \hat{x} + \eta \hat{y} + \zeta \hat{z}) = 0 \]

(74)

that is, for all \( (\xi, \eta) \) the Radon transform of the electric field on the plane \( z = 0 \) is perpendicular to the complex propagation direction \( \xi \hat{x} + \eta \hat{y} + \zeta \hat{z} \) and thus is completely determined from two components of the electric field on the plane \( z = 0 \). The formulas (68)-(74) hold also for the analytic magnetic field provided that \( \bar{E} \) is replaced by \( \bar{H} \). The relation (57) along with the analytic Fourier transform show that for all \( (\xi, \eta) \)

\[ \bar{T}^a(\xi, \eta, t) = \frac{1}{\mu} (\xi \hat{x} + \eta \hat{y} + \zeta \hat{z}) \times \bar{T}^a(\xi, \eta, t) \]

(75)

The analytical Radon transform near-field formulas (66) and (71) are more compact than the corresponding real Radon transform formulas (46) and (50). However, to compute \( T^a \) and \( T^a \) required in (66) and (71) directly form the time-domain measurements of \( \bar{E}(r, t) \) and \( E(r, t) \), one would use (64) or (67) and (72) or (73), respectively. And, as determined above, this procedure is equivalent to using (46) and (50) to compute \( \bar{E}(r, t) \) and \( E(r, t) \), respectively. Thus, the Green’s function representations of Section III-A and the Radon transform formulas obtained from the standard Fourier transform are ultimately the more useful expressions for numerical calculations since they involve only real fields that are obtainable directly from measurements. Moreover, these Radon transform formulas reduce in simplicity to that of the corresponding plane-wave spectrum formulas in the frequency domain when the evanescent fields (second integrals in (46) and (50)) are negligible.

C. Time-Domain Far Fields

This section derives expressions that give the time-domain far fields for \( z > z_0 \) in terms of the time-domain fields on the plane \( z = z_0 \).

Begin by defining a time-domain far-field pattern \( \mathcal{F}(\theta, \phi, t) \) for the acoustic far field such that the acoustic far field is given by \( \Phi(r, t) \sim \mathcal{F}(\theta, \phi, t - r/c)/r \). Then the time-domain far-field pattern is simply the inverse Fourier transform of the frequency-domain far-field pattern defined in Section II-C. The expansion \( R = r - r_0 + \alpha(r^{-1}) \) as \( r \to \infty \) can be inserted into the integrand of (26) to obtain the following expression for the time-domain far-field pattern

\[ \mathcal{F}(\theta, \phi, t) = \frac{\cos \theta}{2\pi \alpha} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\partial \hat{E}}{\partial t} (r_0, t + r_0 \cdot \hat{r}/c) dx_0 dy_0, \]

\[ z > z_0 \quad (76) \]

where \( \hat{r} = \hat{x} \cos \phi \sin \theta + \hat{y} \sin \phi \sin \theta + \hat{z} \cos \theta \) and \( \alpha = x_0^2 + y_0^2 + z_0^2. \)

This result could also have been obtained by Fourier transforming the frequency-domain far field (15). Note that the far field is of order \( r^{-1} \) if the first time derivative of the field on the measurement plane \( z = z_0 \) is a finite function of time.

Similarly, define a time-domain far-field pattern \( \mathcal{F}(\theta, \phi, t) \) for the electric field such that the far electric field is given by \( \bar{E}(r, t) \sim \mathcal{F}(\theta, \phi, t - r/c)/r \). Then one finds from (28) that the time-domain electric far-field pattern is

\[ \mathcal{F}(\theta, \phi, t) = \frac{1}{2\pi c} \bar{E} \times \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \hat{z} \]

(77)

\[ \times \frac{\partial}{\partial t} \bar{E}(r_0, t + r_0 \cdot \hat{r}/c) dx_0 dy_0, \quad z > z_0 \]

which shows that the far electric field is of order \( r^{-1} \) if the first time derivative of the \( x \) and \( y \) components of the electric field on the plane \( z = z_0 \) are finite functions of time. This far-field result was derived previously by Hill [15].

The time-domain magnetic far-field pattern is found from (39) to be

\[ \mathcal{F}^H(\theta, \phi, t) = \frac{1}{2\pi c} \bar{H} \times \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \hat{z} \]

(78)

\[ \times \frac{\partial}{\partial t} \bar{E}(r_0, t + r_0 \cdot \hat{r}/c) dx_0 dy_0, \quad z > z_0 \]

Because the field is zero for \( t < 0 \) the region of integration in (26) is finite, and it is therefore legitimate to insert this expansion for \( R \) into the integrand and then let \( r \to \infty \).
which shows that the far magnetic field is of order $r^{-1}$ if the first time derivative of the $x$ and $y$ components of the electric field on the plane $z = z_0$ are finite functions of time. It also shows that the magnetic far-field pattern equals $\sqrt{\mu/\epsilon} \times \hat{F}$, where $\hat{F}$ is the electric far-field pattern in (77).

The time-domain far fields may also be expressed in terms of the Radon transform of the fields on the plane $z = 0$. Writing the frequency-domain far field (16) in terms of $T_w$ defined in (41), one finds

$$F_w(\theta, \phi) = -\frac{i\omega \cos \theta}{c} e^{-\omega r/c} \cdot T_w (c^{-1} \omega \cos \phi \sin \theta, c^{-1} \omega \sin \phi \sin \theta). \tag{79}$$

Taking the inverse Fourier transform of (79) gives us the time-domain far-field pattern for the acoustic field

$$F(\theta, \phi, t) = \frac{\cos \theta}{c} \frac{\partial}{\partial t} T (c^{-1} \cos \phi \sin \theta, c^{-1} \sin \phi \sin \theta, t) \tag{80}$$

which, upon inserting the definition of $T$ from (42), is seen to equal the far field (76) found from a Green’s function representation.

The electric far-field formula analogous to (80) is

$$\hat{E}(\theta, \phi, t) = \frac{\cos \theta}{c} \frac{\partial}{\partial t} T (c^{-1} \cos \phi \sin \theta, c^{-1} \sin \phi \sin \theta, t) \tag{81}$$

and the magnetic far-field pattern can be expressed from (58) as

$$\hat{H}(\theta, \phi, t) = \sqrt{\mu/\epsilon} \times \hat{E}(\theta, \phi, t) \tag{82}$$

where $\hat{E}$ is the electric far-field pattern in (81). It is easily seen that Radon transform formulas involving analytic fields produce exactly the same far-field formulas presented in (80) and (81).

It should be noted that the far-field formulas of this section are all derived under the assumption that the first time derivatives of the fields are finite functions of time on the plane $z = z_0$. With this assumption the far fields decay as $1/r$ or faster as $r \to \infty$. Also, we prove in [10, App. B] that the electromagnetic far fields decay as $1/r$ (or faster) provided the first time derivative of the source current exists and is bounded by an integrable time-independent function in a finite source region. Nonetheless, classical sources of finite charge and current in a finite region of space can produce classical Maxwellian far-field pulses that decay slower than $1/r$ [32]. However, these anomalous pulses require either infinite acceleration of point charges or infinite first time derivatives of the current [10, sec. 3.3.3]. Moreover, we prove in [10, sec. 3.3.1] that the magnitude of the frequency spectrum of the current (or the velocity of a point charge) located in a finite region of space must decay slower than $|\omega|^{-3/2}$ or $|\omega|^{-1}$ for slower than $1/r$ far-field decay, and must decay as $|\omega|^{-3/2}$ or slower for an "electromagnetic missile" [32] to be produced.

In [10, sec. 3.3.1] it is shown that the time-domain far-field pattern is an analytic function of the observation angles $\theta$ and $\phi$ for bandlimited sources in a finite region of space.

It is also shown that the time-domain far fields of sources, generating fields that can be represented outside the finite source region as a superposition of time-harmonic fields, cannot have zero sidelobes unless the far fields are zero everywhere. Furthermore, it is shown in [10, sec. 3.3.2] that the electromagnetic far fields integrated over all time can be nonzero only if the current remains nonzero as $t \to \infty$. Most of these results for the time-domain far fields also appear in [33].

IV. CONCLUSION

Equations (38) and (39) determine the time-domain electromagnetic fields everywhere to the "right" of the measurement plane directly in terms of integrals of the tangential time-domain field on the measurement plane. These formulas simplify considerably to (77) and (78) when determining the time-domain far fields. (Formulas analogous to (38), (39), (77), and (78) can, of course, be written in terms of integrals of the tangential time-domain magnetic field on the measurement plane.) Alternatively, equations (49)–(59) or (68)–(75), which are analogous to the frequency-domain plane-wave spectrum formulas, determine the time-domain electromagnetic fields to the "right" of the source region indirectly in terms of the time-domain tangential electric field on the measurement plane. In our companion paper (Part 2 of this work) [17], we determine sampling theorems for these spatial integrals of time-dependent functions and compare the efficiency and efficacy of different computation schemes based on these new sampling theorems as well as on conventional sampling theorems in the frequency domain.

REFERENCES


Thorkild B. Hansen (S’91–M’91), for a photograph and biography, please see p. 1402 of the November 1992 issue of this TRANSACTIONS.

Arthur D. Yaghjian (S’68–M’69–SM’84–F’92), for a photograph and biography, please see p. 312 of the March 1992 issue of this TRANSACTIONS.